Online Learning and Game Theory: Regret Lower Bounds and Adaptive Learning Dynamics

Taira Tsuchiya

The University of Tokyo & RIKEN

May 7–8, 2026

ELLIS Milan + RIKEN AIP Joint ELIAS Workshop, Milano

Learning in games

Multiple players interact in a shared environment,
each adapting their strategy to maximize their cumulative reward.

Learning in games

Multiple players interact in a shared environment,
each adapting their strategy to maximize their cumulative reward.

Broader applications

- Minimax optimization, finding equilibria of games
- Variational inequalities

Learning in games

> Multiple players interact in a shared environment,
> each adapting their strategy to maximize their cumulative reward.

Broader applications

- Minimax optimization, finding equilibria of games
- Variational inequalities
- Multi-agent reinforcement learning
- Superhuman AI for poker, Go, Stratego, ...
- Alignment of LLMs

Two-player zero-sum games

- x-player and y-player, each having m_x, m_y actions
- characterized by a **payoff matrix** $A \in [-1, 1]^{m_x \times m_y}$

Two-player zero-sum games

- x-player and y-player, each having m_x, m_y actions
- characterized by a **payoff matrix** $A \in [-1, 1]^{m_x \times m_y}$

Example. Rock-Paper-Scissors

y-player

x-player

			
	$0, 0$	$1, -1$	$-1, 1$
	$-1, 1$	$0, 0$	$1, -1$
	$1, -1$	$-1, 1$	$0, 0$

payoff matrix of the game

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Two-player zero-sum games

- x-player and y-player, each having m_x, m_y actions
- characterized by a **payoff matrix** $A \in [-1, 1]^{m_x \times m_y}$

1. x-player chooses **strategy** $x \in \Delta_{m_x}$ and
   y-player chooses strategy $y \in \Delta_{m_y}$.

(Δ_m: $(m-1)$-dimensional probability simplex)

Two-player zero-sum games

- x-player and y-player, each having m_x, m_y actions
- characterized by a **payoff matrix** $A \in [-1, 1]^{m_x \times m_y}$

1. x-player chooses **strategy** $x \in \Delta_{m_x}$ and
   y-player chooses strategy $y \in \Delta_{m_y}$.
   (Δ_m: $(m-1)$-dimensional probability simplex)
2. x-player gains reward $\langle x, Ay \rangle$, and
   y-player incurs loss $\langle x, Ay \rangle$ (thus zero-sum).

Two-player zero-sum games

- x-player and y-player, each having m_x, m_y actions
- characterized by a **payoff matrix** $A \in [-1, 1]^{m_x \times m_y}$

1. x-player chooses **strategy** $x \in \Delta_{m_x}$ and
   y-player chooses strategy $y \in \Delta_{m_y}$.
   (Δ_m: $(m-1)$-dimensional probability simplex)
2. x-player gains reward $\langle x, Ay \rangle$, and
   y-player incurs loss $\langle x, Ay \rangle$ (thus zero-sum).

Solution of games?
$\rightarrow$ **Nash equilibrium**: a pair of strategies in which no player has an incentive to deviate

Two-player zero-sum games

- x-player and y-player, each having m_x, m_y actions
- characterized by a **payoff matrix** $A \in [-1, 1]^{m_x \times m_y}$

1. x-player chooses **strategy** $x \in \Delta_{m_x}$ and
   y-player chooses strategy $y \in \Delta_{m_y}$.
   (Δ_m: $(m-1)$-dimensional probability simplex)
2. x-player gains reward $\langle x, Ay \rangle$, and
   y-player incurs loss $\langle x, Ay \rangle$ (thus zero-sum).

Solution of games?
$\rightarrow$ **Nash equilibrium**: a pair of strategies in which no player has an incentive to deviate

payoff matrix A is unknown before the game starts
$\rightarrow$ learning in games!

Learning in two-player zero-sum games

A **sequential** formulation of two-player zero-sum games,
characterized by an **unknown** payoff matrix $A \in [-1, 1]^{m_x \times m_y}$

(m_x, m_y: the number of actions of x- and y-players)

At each round $t = 1, \ldots, T$: (Δ_m: $(m-1)$-dimensional probability simplex)

1. x-player selects a strategy $x_t \in \Delta_{m_x}$ and y-player selects $y_t \in \Delta_{m_y}$;

Learning in two-player zero-sum games

A **sequential** formulation of two-player zero-sum games,
characterized by an **unknown** payoff matrix $A \in [-1, 1]^{m_x \times m_y}$

(m_x, m_y: the number of actions of x- and y-players)

At each round $t = 1, \ldots, T$: (Δ_m: $(m-1)$-dimensional probability simplex)

1. x-player selects a strategy $x_t \in \Delta_{m_x}$ and y-player selects $y_t \in \Delta_{m_y}$;
2. x-player gains reward $\langle x_t, Ay_t \rangle$ and
   y-player incurs loss $\langle x_t, Ay_t \rangle$; **(thus zero-sum)**

Learning in two-player zero-sum games

A **sequential** formulation of two-player zero-sum games,
characterized by an **unknown** payoff matrix $A \in [-1, 1]^{m_x \times m_y}$

(m_x, m_y: the number of actions of x- and y-players)

At each round $t = 1, \ldots, T$: (Δ_m: $(m-1)$-dimensional probability simplex)

1. x-player selects a strategy $x_t \in \Delta_{m_x}$ and y-player selects $y_t \in \Delta_{m_y}$;
2. x-player gains reward $\langle x_t, Ay_t \rangle$ and
   y-player incurs loss $\langle x_t, Ay_t \rangle$; **(thus zero-sum)**
3. x-player observes reward vector $g_t = Ay_t$ (**g**ain) and
   y-player observes loss vector $\ell_t = A^\top x_t$;

Learning in two-player zero-sum games

A **sequential** formulation of two-player zero-sum games,
characterized by an **unknown** payoff matrix $A \in [-1, 1]^{m_x \times m_y}$

(m_x, m_y: the number of actions of x- and y-players)

At each round $t = 1, \ldots, T$: (Δ_m: $(m-1)$-dimensional probability simplex)

1. x-player selects a strategy $x_t \in \Delta_{m_x}$ and y-player selects $y_t \in \Delta_{m_y}$;
2. x-player gains reward $\langle x_t, Ay_t \rangle$ and
   y-player incurs loss $\langle x_t, Ay_t \rangle$; **(thus zero-sum)**
3. x-player observes reward vector $g_t = Ay_t$ (**g**ain) and
   y-player observes loss vector $\ell_t = A^\top x_t$;

$\min_x \max_y \langle x, Ay \rangle$

Learning in two-player zero-sum games

A **sequential** formulation of two-player zero-sum games,
characterized by an **unknown** payoff matrix $A \in [-1, 1]^{m_x \times m_y}$

(m_x, m_y: the number of actions of x- and y-players)

At each round $t = 1, \ldots, T$: (Δ_m: $(m-1)$-dimensional probability simplex)

1. x-player selects a strategy $x_t \in \Delta_{m_x}$ and y-player selects $y_t \in \Delta_{m_y}$;
2. x-player gains reward $\langle x_t, Ay_t \rangle$ and
   y-player incurs loss $\langle x_t, Ay_t \rangle$; **(thus zero-sum)**
3. x-player observes reward vector $g_t = Ay_t$ (**g**ain) and
   y-player observes loss vector $\ell_t = A^\top x_t$;

$\min_x \max_y \langle x, Ay \rangle$

Goal of x-/y- **players**: maximize their **cumulative reward** (without knowing A):

- x-player: maximize $\sum_{t=1}^{T} \langle x_t, Ay_t \rangle$,
- y-player: minimize $\sum_{t=1}^{T} \langle x_t, Ay_t \rangle$.

Learning in two-player zero-sum games

A **sequential** formulation of two-player zero-sum games,
characterized by an **unknown** payoff matrix $A \in [-1, 1]^{m_x \times m_y}$

(m_x, m_y: the number of actions of x- and y-players)

At each round $t = 1, \ldots, T$: (Δ_m: $(m-1)$-dimensional probability simplex)

1. x-player selects a strategy $x_t \in \Delta_{m_x}$ and y-player selects $y_t \in \Delta_{m_y}$;
2. x-player gains reward $\langle x_t, Ay_t \rangle$ and
   y-player incurs loss $\langle x_t, Ay_t \rangle$; **(thus zero-sum)**
3. x-player observes reward vector $g_t = Ay_t$ (**g**ain) and
   y-player observes loss vector $\ell_t = A^\top x_t$;

$\min_x \max_y \langle x, Ay \rangle$

Goal of x-/y- players: maximize their **cumulative reward** (without knowing A):

- x-player: maximize $\sum_{t=1}^{T} \langle x_t, Ay_t \rangle$,
- y-player: minimize $\sum_{t=1}^{T} \langle x_t, Ay_t \rangle$.

Deep connection with online learning!

Learning in games as online linear optimization

At each round $t = 1, \ldots, T$: (Δ_m: $(m-1)$-dimensional probability simplex)

1. x-player selects strategy $x_t \in \Delta_{m_x}$ and y-player selects $y_t \in \Delta_{m_y}$;
2. x-player gains reward $\langle x_t, Ay_t \rangle = \langle x_t, g_t \rangle$ and
   y-player incurs loss $\langle x_t, Ay_t \rangle = \langle y_t, \ell_t \rangle$; **(thus zero-sum)**
3. x-player observes reward vector $g_t = Ay_t$ (**g**ain) and
   y-player observes loss vector $\ell_t = A^\top x_t$;

Learning in games as online linear optimization

At each round $t = 1, \dots, T$: (Δ_m: $(m-1)$-dimensional probability simplex)

1. x-player selects strategy $x_t \in \Delta_{m_x}$ and y-player selects $y_t \in \Delta_{m_y}$;
2. x-player gains reward $\langle x_t, Ay_t \rangle = \langle x_t, g_t \rangle$ and
   y-player incurs loss $\langle x_t, Ay_t \rangle = \langle y_t, \ell_t \rangle$; **(thus zero-sum)**
3. x-player observes reward vector $g_t = Ay_t$ (**g**ain) and
   y-player observes loss vector $\ell_t = A^\top x_t$;

Each player solves **online linear optimization over the probability simplex** with regret

- $\mathrm{Reg}_T^x = \max_{x^* \in \Delta_{m_x}} \left\{ \sum_{t=1}^T \langle x^*, g_t \rangle - \sum_{t=1}^T \langle x_t, g_t \rangle \right\}$,
- $\mathrm{Reg}_T^y = \max_{y^* \in \Delta_{m_y}} \left\{ \sum_{t=1}^T \langle y_t, \ell_t \rangle - \sum_{t=1}^T \langle y^*, \ell_t \rangle \right\}$.

No-regret learning dynamics and Nash equilibrium

Theorem (Freund and Schapire 1999)

Let $\bar{x}_T = \frac{1}{T}\sum_{t=1}^{T} x_t$ *and* $\bar{y}_T = \frac{1}{T}\sum_{t=1}^{T} y_t$ *be the average plays.*

$$(\bar{x}_T, \bar{y}_T) \textit{ is a } \frac{\mathrm{Reg}_T^x + \mathrm{Reg}_T^y}{T}\textit{-approximate Nash equilibrium.}$$

No-regret learning dynamics and Nash equilibrium

Theorem (Freund and Schapire 1999)

Let $\bar{x}_T = \frac{1}{T}\sum_{t=1}^{T} x_t$ and $\bar{y}_T = \frac{1}{T}\sum_{t=1}^{T} y_t$ be the average plays.

$$(\bar{x}_T, \bar{y}_T) \text{ is a } \frac{\mathrm{Reg}_T^x + \mathrm{Reg}_T^y}{T}\text{-approximate Nash equilibrium.}$$

Immediate consequence:

$$\mathrm{Reg}_T^x, \mathrm{Reg}_T^y = \widetilde{O}(\sqrt{T}) \quad \Longrightarrow \quad \frac{\mathrm{Reg}_T^x + \mathrm{Reg}_T^y}{T} = \widetilde{O}(1/\sqrt{T}).$$

So $(\bar{x}_T, \bar{y}_T)$ converges to a Nash equilibrium at rate $\widetilde{O}(1/\sqrt{T})$ under uncoupled dynamics.

No-regret learning dynamics and Nash equilibrium

Theorem (Freund and Schapire 1999)

Let $\bar{x}_T = \frac{1}{T}\sum_{t=1}^{T} x_t$ *and* $\bar{y}_T = \frac{1}{T}\sum_{t=1}^{T} y_t$ *be the average plays.*

$$(\bar{x}_T, \bar{y}_T) \textit{ is a } \frac{\mathrm{Reg}_T^x + \mathrm{Reg}_T^y}{T}\textit{-approximate Nash equilibrium.}$$

Immediate consequence:

$$\mathrm{Reg}_T^x, \mathrm{Reg}_T^y = \widetilde{O}(\sqrt{T}) \quad \Longrightarrow \quad \frac{\mathrm{Reg}_T^x + \mathrm{Reg}_T^y}{T} = \widetilde{O}(1/\sqrt{T}).$$

So $(\bar{x}_T, \bar{y}_T)$ converges to a Nash equilibrium at rate $\widetilde{O}(1/\sqrt{T})$ under uncoupled dynamics.

Q. Is this the optimal rate in learning in games?

Fast convergence in games

Hedge algorithm (recall $g_t = Ay_t$ and $\ell_t = A^\top x_t$): $\eta_x, \eta_y \simeq 1/\sqrt{T}$: learning rate

$$x_t(i) \propto \exp\Big(\eta_x \underbrace{\sum_{s=1}^{t-1} g_s(i)}_{\text{cumulative gain of action } i}\Big)\ \forall i \in [m_x], \quad y_t(i) \propto \exp\Big(-\eta_y \underbrace{\sum_{s=1}^{t-1} \ell_s(i)}_{\text{cumulative loss of action } i}\Big)\ \forall i \in [m_y]$$

$\rightarrow \widetilde{O}(1/\sqrt{T})$ convergence rate to Nash equilibrium ☹ ☹

Fast convergence in games

Hedge algorithm (recall $g_t = Ay_t$ and $\ell_t = A^\top x_t$):

$\eta_x, \eta_y \simeq 1/\sqrt{T}$: learning rate

$$x_t(i) \propto \exp\left(\eta_x \sum_{s=1}^{t-1} g_s(i)\right) \forall i \in [m_x], \quad y_t(i) \propto \exp\left(-\eta_y \sum_{s=1}^{t-1} \ell_s(i)\right) \forall i \in [m_y]$$

cumulative gain of action i — cumulative loss of action i

$\rightarrow \widetilde{O}(1/\sqrt{T})$ convergence rate to Nash equilibrium ☹ ☹

Optimistic Hedge algorithm (Rakhlin and Sridharan 2013; Syrgkanis et al. 2015):

$$x_t(i) \propto \exp\left(\eta_x \left(\sum_{s=1}^{t-1} g_s(i) + g_{t-1}(i)\right)\right), \quad y_t(i) \propto \exp\left(-\eta_y \left(\sum_{s=1}^{t-1} \ell_s(i) + \ell_{t-1}(i)\right)\right)$$

Fast convergence in games

Optimistic Hedge algorithm (Rakhlin and Sridharan 2013; Syrgkanis et al. 2015):

$$x_t(i) \propto \exp\left(\eta_x \left(\sum_{s=1}^{t-1} g_s(i) + g_{t-1}(i)\right)\right), \quad y_t(i) \propto \exp\left(-\eta_y \left(\sum_{s=1}^{t-1} \ell_s(i) + \ell_{t-1}(i)\right)\right)$$

Theorem (Rakhlin and Sridharan 2013)

If the x- and y-players follow optimistic Hedge with learning rates $\eta_x = \eta_y = 1/4$, then

$$\mathrm{Reg}_T^x = O(\log(m_x m_y)), \quad \mathrm{Reg}_T^y = O(\log(m_x m_y))$$

which implies an

$$O(\log(m_x m_y)/T) \text{ convergence rate to a Nash equilibrium}.$$

Fast convergence in games

Optimistic Hedge algorithm (Rakhlin and Sridharan 2013; Syrgkanis et al. 2015):

$$x_t(i) \propto \exp\left(\eta_x \left(\sum_{s=1}^{t-1} g_s(i) + g_{t-1}(i)\right)\right), \quad y_t(i) \propto \exp\left(-\eta_y \left(\sum_{s=1}^{t-1} \ell_s(i) + \ell_{t-1}(i)\right)\right)$$

Theorem (Rakhlin and Sridharan 2013)

If the x- and y-players follow optimistic Hedge with learning rates $\eta_x = \eta_y = 1/4$, then

$$\mathrm{Reg}_T^x = O(\log(m_x m_y)), \quad \mathrm{Reg}_T^y = O(\log(m_x m_y))$$

which implies an

$$O(\log(m_x m_y)/T) \text{ convergence rate to a Nash equilibrium}.$$

Rough intuition: If the opponent uses a no-regret algorithm, then we can predict the opponent's next strategy y_{t+1} (and thus gradient $g_{t+1} = Ay_{t+1}$).

Natural questions

Theorem (Rakhlin and Sridharan 2013)

If the x- and y-players fully follow optimistic Hedge with **constant** *learning rates* $\eta_x = \eta_y = 1/4$ *in games with* $A \in [-1, 1]^{m_x \times m_y}$, *then we obtain an*

$$O(\log(m_x m_y)/T)\text{-approximate Nash equilibrium}$$

after T rounds. **Optimal in T!** *(Daskalakis, Deckelbaum, and Kim 2011)*

This result looks great, but ...

Q1 Is the dependence on the number of actions m_x, m_y optimal?

Natural questions

Theorem (Rakhlin and Sridharan 2013)

If the x- and y-players fully follow optimistic Hedge with **constant** *learning rates* $\eta_x = \eta_y = 1/4$ *in games with* $A \in [-1, 1]^{m_x \times m_y}$*, then we obtain an*

$$O(\log(m_x m_y)/T)\text{-approximate Nash equilibrium}$$

after T rounds. **Optimal in T!** *(Daskalakis, Deckelbaum, and Kim 2011)*

This result looks great, but ...

Q1 Is the dependence on the number of actions m_x, m_y optimal?

Q2 What if the opponent deviates from optimistic Hedge with a constant learning rate?

Natural questions

Theorem (Rakhlin and Sridharan 2013)

If the x- and y-players fully follow optimistic Hedge with **constant** *learning rates $\eta_x = \eta_y = 1/4$ in games with $A \in [-1, 1]^{m_x \times m_y}$, then we obtain an*

$$O(\log(m_x m_y)/T)\text{-approximate Nash equilibrium}$$

after T rounds. **Optimal in T!** *(Daskalakis, Deckelbaum, and Kim 2011)*

This result looks great, but ...

Q1 Is the dependence on the number of actions m_x, m_y optimal?

Q2 What if the opponent deviates from optimistic Hedge with a constant learning rate?

Q3 The algorithm assumes a known payoff scale, $A \in [-1, 1]^{m_x \times m_y}$. Can we achieve fast convergence without knowing the scale?

Natural questions

Theorem (Rakhlin and Sridharan 2013)

If the x- and y-players fully follow optimistic Hedge with **constant** *learning rates* $\eta_x = \eta_y = 1/4$ *in games with* $A \in [-1, 1]^{m_x \times m_y}$*, then we obtain an*

$$O(\log(m_x m_y)/T)\text{-approximate Nash equilibrium}$$

after T *rounds.* **Optimal in** T**!** *(Daskalakis, Deckelbaum, and Kim 2011)*

This result looks great, but ...

Q1 Is the dependence on the number of actions m_x, m_y optimal?
[Tsuchiya, AISTATS2026 Spotlight]

Q2 What if the opponent deviates from optimistic Hedge with a constant learning rate?
[Tsuchiya–Ito–Luo, COLT2025]

Q3 The algorithm assumes a known payoff scale, $A \in [-1, 1]^{m_x \times m_y}$. Can we achieve fast convergence without knowing the scale? [Tsuchiya–Luo–Ito, 2026]

Q1. Optimal dependence on the number of actions?

(recap) Optimistic Hedge with constant learning rates (Rakhlin and Sridharan 2013) gives

$O(\log(m_x m_y)/T)$-approximate Nash equilibrium

Q1. Optimal dependence on the number of actions?

(recap) Optimistic Hedge with constant learning rates (Rakhlin and Sridharan 2013) gives

$$O(\log(m_x m_y)/T)\text{-approximate Nash equilibrium}$$

Main results (informal, [T. AISTATS'26, Spotlight]):

- This can be improved to

$$O(\sqrt{\log m_x \log m_y}/T) .$$

→ Particularly important when m_x and m_y are imbalanced

Q1. Optimal dependence on the number of actions?

(recap) Optimistic Hedge with constant learning rates (Rakhlin and Sridharan 2013) gives

$$O(\log(m_x m_y)/T)\text{-approximate Nash equilibrium}$$

Main results (informal, [T. AISTATS'26, Spotlight]):

- This can be improved to
$$O(\sqrt{\log m_x \log m_y}/T)\,.$$
→ Particularly important when m_x and m_y are imbalanced
- This is optimal for optimistic Hedge with constant learning rates:
$$\mathrm{Reg}_T^x = \Omega(\sqrt{\log m_x \log m_y})\,, \quad \mathrm{Reg}_T^y = \Omega(\sqrt{\log m_x \log m_y})\,.$$

Q1. Optimal dependence on the number of actions?

(recap) Optimistic Hedge with constant learning rates (Rakhlin and Sridharan 2013) gives

$$O(\log(m_x m_y)/T)\text{-approximate Nash equilibrium}$$

Main results (informal, [T. AISTATS'26, Spotlight]):

- This can be improved to

$$O(\sqrt{\log m_x \log m_y}/T)\,.$$

  → Particularly important when m_x and m_y are imbalanced

- This is optimal for optimistic Hedge with constant learning rates:

$$\mathrm{Reg}_T^x = \Omega(\sqrt{\log m_x \log m_y})\,, \quad \mathrm{Reg}_T^y = \Omega(\sqrt{\log m_x \log m_y})\,.$$

  → (Partially) resolves a question raised by Anagnostides–Kalavasis–Sandholm–Zampetakis (2024) on the optimal dependence on the number of actions for optimistic Hedge.

Q1. Optimal dependence on the number of actions?

Also in the paper

- Refined upper bound via convex reformulation over (η, η', c, c').
- Dynamic regret (connection with last-iterate convergence): near-matching $\sqrt{\log m_x \log m_y} \cdot \log T$ regret upper and lower bounds.
- Extension to Hölder-smooth convex–concave games.

Q1. Optimal dependence on the number of actions?

Also in the paper

- Refined upper bound via convex reformulation over (η, η', c, c').
- Dynamic regret (connection with last-iterate convergence): near-matching $\sqrt{\log m_x \log m_y} \cdot \log T$ regret upper and lower bounds.
- Extension to Hölder-smooth convex–concave games.

Future work

- Learning-rate-independent lower bounds for optimistic Hedge.
- Algorithm-independent lower bounds for general strongly-uncoupled dynamics.

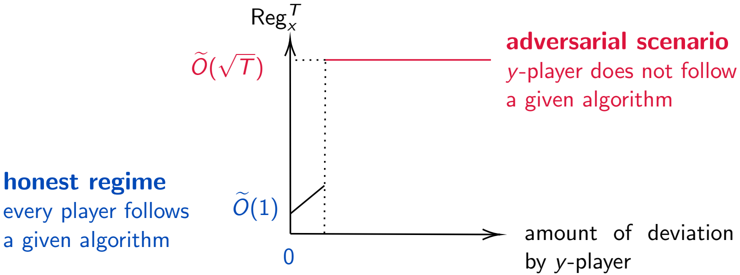
Q2. What if the opponent deviates?

Known solution (Syrgkanis et al. 2015): Monitor gradient variation $\sum_{s=1}^{t-1} \|g_s - g_{s+1}\|_1^2$, and if it exceeds a threshold, switch to an algorithm with a worst-case regret of $\widetilde{O}(\sqrt{T})$

Q2. What if the opponent deviates?

Known solution (Syrgkanis et al. 2015): Monitor gradient variation $\sum_{s=1}^{t-1}\|g_s - g_{s+1}\|_1^2$, and if it exceeds a threshold, switch to an algorithm with a worst-case regret of $\widetilde{O}(\sqrt{T})$

Discontinuous behavior: A slight deviation of the y-player from a given algorithm can suddenly cause the x-player to suffer a regret of $O(\sqrt{T})$ ☹ ☹



Q2. What if the opponent deviates? (cont'd)

Main results (informal, [T.–Ito–Luo COLT'25]):

There exists a learning dynamic robust against deviations by the opponent. In particular,

- Show that there exists a learning dynamic such that

$$\mathrm{Reg}_T^x = \widetilde{O}\left(\sqrt{1 + C_y} + C_x\right), \quad \mathrm{Reg}_T^y = \widetilde{O}\left(\sqrt{1 + C_x} + C_y\right).$$

y-player's cumulative deviation (C_y)

x-player's cumulative deviation (C_x)

Q2. What if the opponent deviates? (cont'd)

Main results (informal, [T.–Ito–Luo COLT'25]):

There exists a learning dynamic robust against deviations by the opponent. In particular,

- Show that there exists a learning dynamic such that

$$\mathrm{Reg}_T^x = \widetilde{O}\left(\sqrt{1+C_y} + C_x\right), \quad \mathrm{Reg}_T^y = \widetilde{O}\left(\sqrt{1+C_x} + C_y\right).$$

C_y: y-player's cumulative deviation

C_x: x-player's cumulative deviation

- Show that the above bounds are optimal

$$\mathrm{Reg}_T^x = \Omega\left(\sqrt{1+C_y} + C_x\right), \quad \mathrm{Reg}_T^y = \Omega\left(\sqrt{1+C_x} + C_y\right).$$

Q3. Is scale-free fast convergence possible?

Scale-free learning dynamic (informal):

- no prior knowledge of the payoff scale is needed; and
- if payoffs are rescaled by any constant $c > 0$, the strategy sequence is unchanged.
  (A key property behind the success of regret matching!)

Q3. Is scale-free fast convergence possible?

Scale-free learning dynamic (informal):

- no prior knowledge of the payoff scale is needed; and
- if payoffs are rescaled by any constant $c > 0$, the strategy sequence is unchanged. (A key property behind the success of regret matching!)

Main result (informal, from [T.–Luo–Ito '26]):

> There exists a scale-free learning dynamic such that for any payoff matrix $A \in [-A_{\max}, A_{\max}]^{m_x \times m_y}$, it gives
>
> $$\widetilde{O}(A_{\max}/T)\text{-approximate Nash equilibrium}$$
>
> after T rounds.

Q2, Q3

More results in the papers

Extension to multiplayer general-sum games

$\rightarrow$ Fast convergence to a correlated equilibrium via swap regret

Q2, Q3

More results in the papers

Extension to multiplayer general-sum games

$\rightarrow$ Fast convergence to a correlated equilibrium via swap regret

Technique

For Q2:

- stability analysis of Markov-chain stationary distributions for optimistic FTRL with adaptive learning rates
- (dealing with adversarial corruption of the gradient feedback)

Q2, Q3

More results in the papers

Extension to multiplayer general-sum games

→ Fast convergence to a correlated equilibrium via swap regret

Technique

For Q2:

- stability analysis of Markov-chain stationary distributions for optimistic FTRL with adaptive learning rates
- (dealing with adversarial corruption of the gradient feedback)

For Q3:

- A new stopping-time analysis to exploit negative terms without knowing the scale
- **doubling clipping**: A new scale-free learning algorithm for online linear optimization useful in swap regret minimization

Summary

Takeaway

Fast convergence in learning in games can be made
dimension-efficient • deviation-robust • scale-free

Thanks for listening!

References I

- Daskalakis, Constantinos, Alan Deckelbaum, and Anthony Kim (2011). "Near-optimal no-regret algorithms for zero-sum games". In: *Proceedings of the Twenty-Second Annual ACM-SIAM Symposium on Discrete Algorithms*. Society for Industrial and Applied Mathematics, pp. 235–254.
- Freund, Yoav and Robert E. Schapire (1999). "Adaptive Game Playing Using Multiplicative Weights". In: *Games and Economic Behavior* 29.1, pp. 79–103.
- Rakhlin, Sasha and Karthik Sridharan (2013). "Optimization, Learning, and Games with Predictable Sequences". In: *Advances in Neural Information Processing Systems*. Vol. 26. Curran Associates, Inc., pp. 3066–3074.
- Syrgkanis, Vasilis et al. (2015). "Fast Convergence of Regularized Learning in Games". In: *Advances in Neural Information Processing Systems*. Vol. 28. Curran Associates, Inc., pp. 2989–2997.